Sorites Hourglass

I believe that the paradoxical component of Sorites Paradox arises in our logics inability to fully capture the underlying realities of vagueness across a continuum of small changes – and that to properly resolve the changing of a predicate over a soritical progression we need to develop different ways of handling precision in our logical systems. I propose to show this belief by modelling an hourglass in a ** Łε** logical system, and then show how we can potentially handle resolving the paradox without a sharp cut-off point.

To begin with, it is useful to work from a simple induction of a soritical progression.

Let us represent a pile of sand by a and the amount of sand in said pile as its index, ie, a_1 being one grain of sand, a_2 being two grains and so on.

Let us represent the "heapedness" of the pile as predicate H, assigning truth in H to the pile of sand being a "heap".

We can plainly say for instance, that at a sufficiently large number of grains of sand, m, we have a heap of sand. Therefore, Ha_m.

We can then say, that if we have a heap of sand, if we took away one grain of sand it would still be a heap. This is the implication that is at the heart of the Sorites Paradox. To express it formally, we induce the following;

Ha_m

$Ha_m \supset Ha_{m-1}$

$\Rightarrow (\forall n)\ Ha_n \supset Ha_{n-1}$

$\Rightarrow (\forall n)\ Ha_n$

I believe that the main issue with Sorites Paradox springs from the universalised implication as shown above. However, it is reasonable to say "if we have a heap of sand, if we took away one grain of sand it would still be a heap", and $(\forall n)\ Ha_n \supset Ha_{n-1}$ is as close to expressing those semantics in formal language as is possible. And in and of itself this is not paradoxical either. Without a contradictory condition, such as "one grain of sand is not a heap", ie, $\neg Ha_0$, all we have is a predicate that has been universalised by substitution of a soritical progression. It is that we expect the predicate to become false over the course of the progression of implications that we have a paradox. This expectation really does need to be formalised to discuss it, hence;

$v(Ha_0) = 0,\ v(Ha_m) = 1$

By formally declaring our boundaries, we are able to discuss the soritical progression without having to deal with further problems of vagueness or introducing problems involving infinities.

This is where the inability to fully model reality in our logics becomes apparent – in that we have essentially placed truth and falsehood at either ends of a continuum, one which we have a valid universal implication of truth already established by a series of substitutions. In order to resolve this, we need to look at each a_n and investigate the relationship with the surrounding neighbours, ie, a_{n-1} and a_{n+1}. To begin with, we see that $(\forall n)\ Ha_n \supset Ha_{n-1}$ clearly establishes a relationship between neighbouring elements. This implication arises from our reasonable semantic argument that a single grain of sand is not enough of a difference to change the heapedness of our pile of sand. What follows on from this is that a change of a single grain of sand is indistinguishable from its neighbours. While on one hand we are acknowledging that there is a change, the other hand claims that change to be indistinguishable; that is to say, that two piles which differ by a single grain of sand are equivalent and by applying substitution in the same manner as we universalised the original implication, we reach the concept that **all** changes are indistinguishable and therefore equivalent, whether they be one grain of sand or two or ten. An attempt at formalising this would be;

$a_m \equiv a_{m-1}$

$\Rightarrow (\forall n)\ a_n \equiv a_{n-1}$

By formalising with equivalence, we directly take on what I believe to be the inability to model reality I have previously alluded to. We can see this through one objection to this argument, which would be that $(\forall n)\ Ha_n \supset Ha_{n-1} \Rightarrow (\forall n)\ a_n \equiv a_{n-1}$ is invalid. And it is. But that isn't the basis of our argument that changes are equivalent; our basis is the same underlying reality that gives rise to the implication of heapedness in the first place. The paradoxical nature of the equivalence of all sizes of a pile of sand are immediately evident; we would get $a_m \equiv a_0$ which is clearly invalid as these are true and false respectively. This result must give us pause to examine our assumptions of reality that create Sorites Paradox.

To begin with, we must reject the equivalence of $a_m \equiv a_{m-1}$ and look for a logical system which will adequately model and satisfy $(\forall n)\ a_n \not\equiv a_{n-1} \models (\forall n)\ Ha_n \supset Ha_{n-1}$. Then we must look as to why this logical system is capable of resolving Sorites Paradox, with reference to the ability to model vagueness as experienced in real situations.

I believe that Ƚε is the system we are looking for; it allows us to express the change of a_n to a_{n-1} by a change in $v(Ha_{n-1})$, which should be sufficient to represent the gradual accumulation of small changes. And it satisfies that we can have the heapedness implicated between neighbouring elements without those elements being equivalent.

In order to gauge how well Ƚε is able to model vagueness as experienced in real situations, it will be useful to look at Sorites Paradox from the other direction; that is, from having one grain of sand that is not a heap, and then implying upwards that adding a grain of sand does not make our not heap of sand a heap, as induced thusly;

$\neg Hb_1$

$\neg Hb_1 \supset \neg Hb_2$

$\Rightarrow (\forall n)\ \neg Hb_n \supset \neg Hb_{n+1}$

$\Rightarrow (\forall n)\ \neg Hb_n$

$v(Hb_0) = 0,\ v(Hb_m) = 1.$

In fact, we are able to show ***a & b*** in tandem, with $a_m = b_0$ and $a_0 = b_m$ ie, modelling an hourglass with ***a*** being the upper half with sand flowing into ***b*** at the lower half. By using this as a model, we are able to keep a constant amount of sand, ***m*** and have a clearer ability to distinguish large changes in ***a*** or ***b*** by observing changes in complements. We do need to formalise the relationship between ***a*** and ***b***, which is; $(\forall n)\ a_n \equiv b_{m-n}$

We also get the interesting dynamic that at $n = m/2$, ***a*** and ***b*** will be the same, with ***Ha*** and ***¬Hb***. This is not to say that $n = m/2$ is a turning point between ***H*** and ***¬H***, but to illustrate that Sorites Paradox works by taking a predicate and extending it across an entire continuum, and the paradoxical nature thereof. In fact, it could be valid to say that at the point where ***a=b*** both ***a*** and ***b*** are heaps – it certainly would be valid to say that when $a=b$, $v(Ha) = v(Hb)$.

Interesting dynamics aside, this helps us to visualise and distinguish all of the points between ***0*** and ***m***, which helps us in analysing our logics ability to model the underlying reality. Observe the following chart;

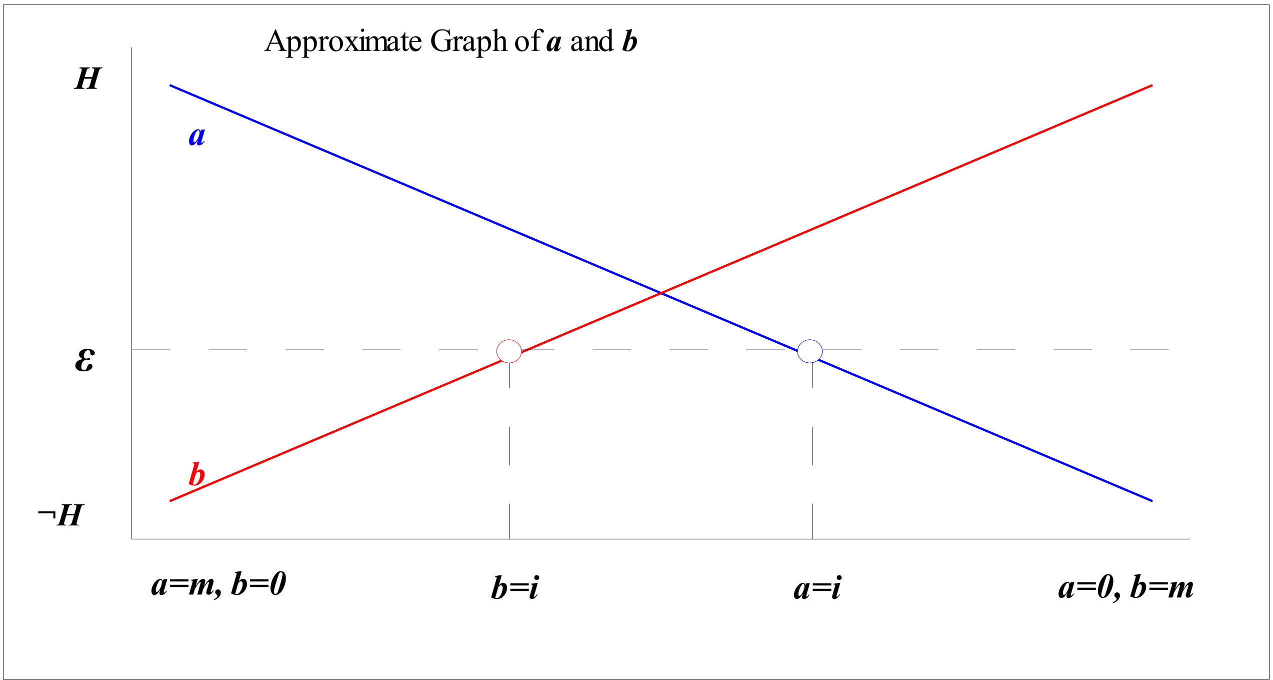

Figure 1

Figure 1 is merely an approximation and should only be referred to as a diagrammatic representation of what is going on. For example, I would not consider the relationship between n and $v(Ha_n)$ or $v(Hb_n)$ to be linear. In reality, it would make much more sense for them to be non-linear, with their slope changing closer to points ***0, i and m.***

One point of note though, is the point of i and it's relation to ε. As shown on the chart, point i is not considered to be part of the continuum, but a limit thereof. This is the means through which I intend to show a solution to Sorites Paradox without a sharp "cut-off" point. This seems highly counter-intuitive, but that the main argument against sharp cut-offs is their counter-intuitive nature.

I derive this from the fact that the continuum of values between H and $\neg H$ is (0,1), ie, a continuum of real numbers. We can then treat them accordingly and consider the problem as a mathematical as well as a logical dilemma. From there, it is a small step to considering limits as a solution, specifically working a solution as a limit within a function.

Now, a diagram is not a proof, so let's construct a formal definition of this concept;

$v(Ha_{0<n<i}) < \varepsilon,\ v(Ha_{i<n<m}) > \varepsilon$

That is to say, that at point i the heapedness of a is indeterminate; and is determined for values above and below. Also note that i is directly related to ε. This, combined with the fact that there are an infinite number of Ƚε logics, allows for an infinite number of cut-off points. I believe that with this, the question of resolving a particular Sorites Paradox lies not in arguing whether the cut-off point exists or not, but in solving the underlying reality for a function of truth and then to find point i and with it the designated value ε.

I defend this belief by appealing to an observation of an hourglass – by having a constant amount m shifting through two changing progressive variables, we are able to better distinguish changes within the progression, as each change of n plus or minus one is seen in both a and b. Thusly, we can observe the approximate point where we would consider H to go from true to false. However, it is difficult to pinpoint exactly when this point occurs, but we can easily discuss when a pile within our hourglass is approaching or has moved past this point. By substitution we can then solve for the location of this point, and then assign it as a limit.

Furthermore, as we are dealing in a function that maps integers of n into real numbers of $v(H)$, we have room to deal with precision in that mapping. We can "soften" the change of point i through clever use of approaching the limit ε in real numbers versus how we approach the limit i.

Thusly, I think that we have shifted the burden of the cut-off point from proving and maintaining its existence to finding a general solution for a particular Sorites Paradox. By embracing the counter-intuitive nature of cut-off points with mathematical tools designed to deal with turning points and shifts within continuous functions, we are able to work on a different level to resolve the issues surrounding a transition of a vague predicate.

A criticism could be levelled that this is merely an *a priori* exercise in adding another layer to an already vague and confusing Paradox, which would of course be a valid criticism and one worthy of further discussion. However, one must take into account that other proposed solutions to Sorites Paradox, such as super-valuation, are also capable of the same criticism.

In conclusion, I believe that a **Łε** logic where one solves a Sorites Paradox for the designated value through crafting a function that maps the underlying reality to truth values and determines a limiting point where the truth value changes around, is one of the strongest options for resolutions of different variants of Sorites Paradox. This allows us to bring to bear work within semantic, formal logic, and mathematical tools to the paradox, an approach which may well be stronger than relying on one branch alone.

References

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